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Series expansion of the percolation probability for the directed square lattice

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Abstract. By extrapolation from finite lattices, we extend the known series for the percolation probability on the directed square lattice from eight terms to 41. Analysing the series, we obtain the estimates $q_c = 0.355\,299 \pm 0.000\,001$, $\beta = 0.2764 \pm 0.0001$ for the critical probability and the critical exponent. From this, together with scaling relations and previous results on the moments of the pair-connectedness function, we conjecture that β may be exactly $\frac{199}{720}$.

1. Introduction

The bond percolation problem on the directed square lattice has been discussed by Blease (1977), Kinzel (1983) and De'Bell and Essam (1983). More recently, Essam *et al* (1986) extended the mean size and second moment of the pair-connectedness function series to 35 terms, and in a forthcoming paper Essam *et al* (1988) report on the extension of these series to 49 terms, as well as corresponding series for the directed triangular lattice bond and site problems, and the directed square lattice site problem. However, for the percolation probability, the longest previously published series has remained the eight-term series of Blease.

We have extended the percolation probability series to 41 terms. In the following section we show how we have done this by calculating the probability for finite lattices and then extrapolating from these. In § 3 we report on the analysis of this series, and conclude that $\beta = 0.2764 \pm 0.0001$. This is consistent with the rational value $\beta = \frac{199}{720} = 0.276\,3888\ldots$. In Essam *et al* (1988) the conjectured values of other exponents are

$$\gamma = \tfrac{41}{18} \qquad \nu_\perp = \tfrac{79}{72} \qquad \nu_\parallel = \tfrac{26}{15}. \tag{1.1}$$

Scaling then gives the rational value of β we suggest. The aesthetically more appealing value $\beta = \frac{200}{720} = \frac{5}{18} = 0.2777\ldots$ is excluded by the numerical evidence.

2. Finite-lattice calculations

Consider a square lattice, drawn diagonally as in figure 1. Place bonds on the edges so that each edge has probability p of containing a bond and probability $q = 1 - p$ of being empty. Regard two sites as connected if and only if an observer can walk along bonds from the upper to the lower always going downwards (either down and to the

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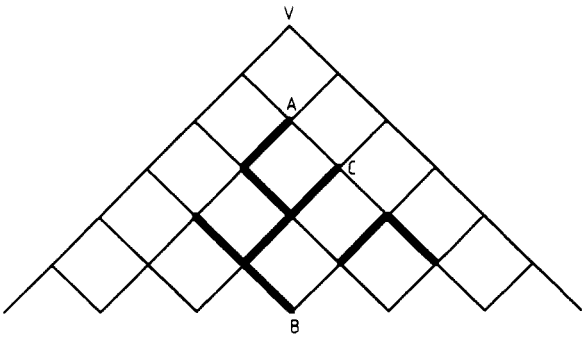

Figure 1. The square lattice L, drawn diagonally, showing an arrangment of bonds on its edges.

right, or down and to the left). Thus A and B in figure 1 are connected, as are C and B, while A and C are not connected.

For the infinite system, when q is less than some critical value q_c, there is an infinite cluster; and there is a non-zero probability $P(q)$ that a given site V is connected to that cluster by a path that runs downwards from V.

Note that such a path can only lead to points that lie below V, no further to the left nor the right than they are below V. This suggests a finite-lattice analogue of $P(q)$, namely to consider a pyramid-shaped lattice L of N rows, the top row having one site, the next two sites, and so on, as in figure 1. Thus there are N sites in the bottom row and $N(N+1)/2$ sites altogether. Let V be the site at the apex of L, and define

$$P_N(q) = \text{probability that V is connected to at least one site in the bottom row.} \tag{2.1}$$

Then for $q < q_c$ we expect that

$$P(q) = \lim_{N\to\infty} P_N(q). \tag{2.2}$$

In fact we can take this as a more precise definition of $P(q)$.

2.1. Ising model formulation

We can express $P_N(q)$ as the partition function of an Ising model on L, with one-, two- and three-spin interactions. With each site of L associate a state (or 'spin') σ_j such that

$$\sigma_j = \begin{cases} +1 & \text{if site } j \text{ is connected to at least one site in the bottom row} \\ -1 & \text{otherwise.} \end{cases} \tag{2.3}$$

We shall write $+1, -1$ simply as $+, -$.

Let k, l be the sites immediately below j, as in figure 2. For σ_j to be $+1$, j must be linked to k (or l) by a bond, and σ_k (or σ_l) must be $+1$.

Figure 2. A site j of L and two lower adjacent sites k, l. The four possible arrangements of bonds on the two edges are shown. If $\sigma_k = -$ and $\sigma_l = +$, then σ_j has the values indicated. Thus $W(-|-,+) = q^2 + pq = q$; $W(+|-,+) = pq + p^2 = p = 1 + q$.

We can therefore define a function

$$W(\sigma_j \mid \sigma_k \sigma_l) = \text{probability that site } j \text{ is in state } \sigma_j, \text{ given that sites } k, l \text{ are in states } \sigma_k, \sigma_l. \tag{2.4}$$

Enumerating the various arrangements of states and of bonds on the edges (j, k), (j, l), some of which are shown in figure 2, we obtain

$$W(-\mid b, c) = q^{(b+c+2)/2} \qquad W(+\mid b, c) = 1 - q^{(b+c+2)/2}. \tag{2.5}$$

Let $f_1(\sigma_1)$ be the probability that the apex V of L is in state σ_1 and let $f_2(\sigma_1', \sigma_2')$ be the probability that the two sites in the row below V are in states σ_1', σ_2', respectively. Then it follows that

$$P_N(q) = f_1(+) \tag{2.6}$$

$$f_1(\sigma_1) = \sum_{\sigma_1'} \sum_{\sigma_2'} W(\sigma_1 \mid \sigma_1', \sigma_2') f_2(\sigma_1', \sigma_2'). \tag{2.7}$$

More generally, if $f_r(\sigma_1, \ldots, \sigma_r)$ is the probability that the sites in row r (counting downwards from the top) are in states $\sigma_1, \ldots, \sigma_r$, then

$$f_r(\sigma_1, \ldots, \sigma_r) = \sum_{\sigma_1', \ldots, \sigma_{r+1}'} \left(\prod_{j=1}^{r} W(\sigma_j \mid \sigma_j', \sigma_{j+1}') \right) f_{r+1}(\sigma_1', \ldots, \sigma_{r+1}') \tag{2.8}$$

for $r = 1, 2, \ldots, N-1$.

These relations together with the boundary condition

$$f_N(\sigma_1, \ldots, \sigma_N) = \prod_{j=1}^{N} \delta(\sigma_j, +) \tag{2.9}$$

define the functions f_r. They can be thought of as transfer matrix relations (see, e.g., Baxter (1982, ch 3, 6)) that build up the lattice from the bottom to the top, and are the same as those of Bidaux and Forgacs (1984). Combining them, we obtain

$$f_1(\sigma_1) = \sum_{\{\sigma\}} \prod_j W(\sigma_j \mid \sigma_k, \sigma_l) \tag{2.10}$$

where the product is over all sites j of L that are above the bottom row. For each site j, k and l are the two sites below it, as in figure 2. The sum is over all values ± 1 of each σ_j, other than the topmost spin σ_1. The spins in the bottom row are fixed to be $+1$.

Thus $f_1(\sigma_1)$ is the partition function of an Ising model on L, with interaction function $W(\sigma_j \mid \sigma_k, \sigma_l)$ for each up-pointing triangle j, k, l, and fixed spins at top and bottom, as indicated in figure 3.

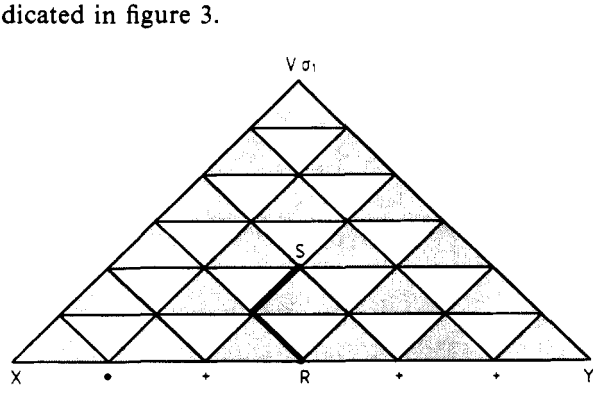

Figure 3. The lattice L, showing the shaded triangles in which there is an interaction weight function $W(\sigma_j \mid \sigma_{k_1}, \sigma_l)$.

From (2.5),

$$\sum_a W(a\,|\,b, c) = 1 \tag{2.11}$$

for all values of b, c. Summing (2.8) over $\sigma_1, \ldots, \sigma_r$, it follows that

$$\sum_{\sigma_1,\ldots,\sigma_r} f_r(\sigma_1, \ldots, \sigma_r) = 1 \tag{2.12}$$

for $r = N, N-1, \ldots, 1$, so

$$f_1(+) + f_1(-) = 1. \tag{2.13}$$

The expression $f_1(+)+f_1(-)$ is just the partition function Z when the top spin σ_1 is free to take either value, so we see that Z is trivially equal to 1. We are therefore in a rather unusual position where the partition function Z does not grow or decay exponentially with the number of sites, but stays fixed: in fact we have an Ising model at a disorder point (Baxter 1984 and references therein). Similarly, we expect $P_N(q) = f_1(+)$ to tend to a limit as N becomes large.

The formulation (2.10) is adequate for our purposes, but it is an unusual Ising model in that it contains three-spin interactions on the shaded triangles. In fact we can express it as a conventional model on a honeycomb lattice, with only one- and two-spin interactions between adjacent spins. To do this, note that (2.5) can be written as

$$W(a\,|\,b, c) = \sum_{d=\pm} f_{ad} g_{bd} g_{cd} \tag{2.14}$$

where

$$\begin{aligned} &f_{++} = -1 \qquad f_{+-} = f_{-+} = 1 \qquad f_{--} = 0 \\ &g_{++} = q \qquad g_{+-} = g_{-+} = g_{--} = 1. \end{aligned} \tag{2.15}$$

This means that the triangle weight function W is the same as that of a three-pointed star with centre spin d (which is summed over) and corner spins a, b, c. Replacing each shaded triangle in figure 3 by such a star, we obtain the honeycomb lattice of figure 4. Hence $f_1(\sigma_1)$ is the partition function of a model on this lattice, with weight function $f(\sigma_i, \sigma_j)$ for vertical edges (i, j) and $g(\sigma_i, \sigma_j)$ for non-vertical edges. This is a special case of the anisotropic honeycomb lattice Ising model in a magnetic field.

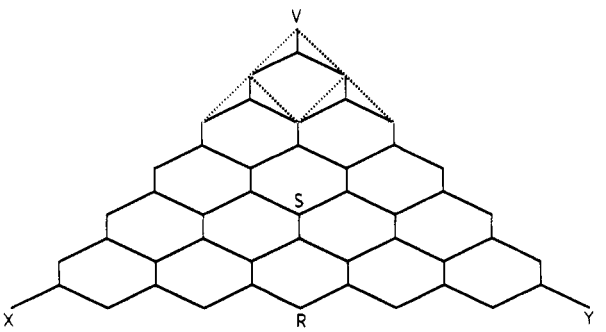

Figure 4.

We can transform from this honeycomb lattice model back to a triangular model by summing over the spins at the top of vertical edges. This gives a model rather like the original one of figure 3, but now with interactions within down-pointing triangles (the unshaded triangles of figure 3). This appears to be the end of this sequence of transformations; unlike the zero-field Ising model, which can be solved by successive triangle-to-star and star-to-triangle transformations (Baxter and Enting 1978, Hilhorst *et al* 1978).

2.2. Calculation technique

For small N one can calculate $P_N(q)$ by successively using (2.8) and (2.9) to calculate $f_{N-1}, f_{N-2}, \ldots, f_1$; finally using (2.6). One finds that

$$P_1(q)=1 \qquad P_2(q)=1-q^2 \qquad P_3(q)=1-q^2-2q^3+q^4+2q^5-q^6$$
$$P_4(q)=1-q^2-2q^3-4q^4+6q^5+11q^6-10q^7-10q^8+10q^9+2q^{10}-4q^{11}+q^{12}. \tag{2.16}$$

We wrote a FORTRAN program to do this to a given order in an expansion in powers of q. The method is basically a transfer matrix method and works well, but is limited by storage, which grows as 2^N. To alleviate this problem, we introduced a cut near the lower centre of the lattice: the bold line RS in figure 3. We fixed the spins on this cut and used the transfer matrix technique to build up the lattice clockwise round S, starting from XR, working upwards to VS, then downwards to YR. This was repeated for all values of the spins on the cut, and the answer for $f_1(+)$ was summed over all such repetitions. This means that storage grew only as $2^{N/2}$, though time continued to grow as 2^N.

The coefficients of the q-series expansions of $f_r(\sigma_1, \ldots, \sigma_r)$ are integers and grow quite rapidly with order. We avoided integer overflow by using modular (or residue) arithmetic (see chapter 13 of Young and Gregory (1973)). The calculations were performed twice, with moduli 10^8 and 10^8+1, respectively: this enables us to handle integers as large as 10^{16}. (Since division was not required, there was no difficulty in using non-prime moduli.)

Using these methods, one of us (RJB) was able to calculate $P_N(q)$ for $N=1, \ldots, 24$ to order $N+13$, using only an Apple MacIntosh. The other (AJG) continued the calculation to $N=29$ on an ELXSI and a Microvax II.

Unlike other two-dimensional models, we were unable to put it in a form where we could use the powerful corner transfer matrix technique (Baxter and Enting 1979, Baxter *et al* 1980). Even so, we have been able to extend the series to considerable length by using the following extrapolation method.

2.3. Extrapolation

Let

$$P_N(q)=\sum_{m=0}^{\infty} a_{Nm} q^m. \tag{2.17}$$

Then we always found that

$$a_{Nm}=a_{mm} \qquad N \geqslant m \tag{2.18}$$

i.e. going from N to $N+1$ leaves the coefficients of $1, q, \ldots, q^N$ unchanged. This is consistent with the expectation that $P_N(q)$ tends to the limit $P(q)$, and implies that

$$P(q) = P_\infty(q) = \sum_{N=0}^{\infty} a_{NN} q^N. \tag{2.19}$$

Thus our finite-lattice results for $1 \leqslant N \leqslant 29$ immediately give us the first 29 coefficients in the series expansion of $P(q)$.

We were able to significantly improve on this. Let

$$d_N^r = a_{N,N+r} - a_{N+1,N+r}. \tag{2.20}$$

Then from (2.18)

$$d_N^r = 0 \qquad r \leqslant 0 \tag{2.21}$$

while d_N^1 is the difference between $a_{N,N+1}$ and the limiting $(N+1)$th-order coefficient $a_{N+1,N+1}$. Thus $d_N^1 q^{N+1}$ is the leading error of $P_N(q)$, considered as an approximation to $P(q)$. We observed that

$$d_1^1, d_2^1, d_3^1, \ldots = 1, 2, 5, 14, 42, 132, \ldots. \tag{2.22}$$

These are the Catalan numbers (Sloane 1973)

$$c_N = (2N)!/N!(N+1)! \tag{2.23}$$

which also occur elsewhere in lattice statistics (Baxter 1987).

Indeed we find, for all the available values $N = 1, \ldots, 28$, that

$$d_N^1 = c_N. \tag{2.24}$$

We assume, as seems eminently reasonable, that this holds for all N. In fact, one referee of this paper has pointed out that this can be proved to be so by counting appropriate subsets of compact directed animals (cf Bhat *et al* 1986, Forgacs and Privman 1987). We can therefore use our $N = 29$ results to calculate $a_{30,30}$, and hence extend the series for $P(q)$ by one term.

The first few numbers d_N^2 are

$$d_1^2, d_2^2, \ldots = 0, -1, -4, -14, -48, -165, \ldots.$$

This sequence is also in Sloane (1973):

$$d_N^2 = 2c_N - c_{N+1}. \tag{2.25}$$

Again, we find that this is true for all our available values $N = 1, \ldots, 28$. Extrapolating and taking (r, N) in (2.20) to be $(2, 29)$, $(1, 30)$, we can successively calculate $a_{30,31}$, $a_{31,31}$, and hence extend our series for $P(q)$ by one more term.

Since

$$c_N/c_{N-1} = (4N-2)/(N+1) \tag{2.26}$$

these results for d_N^1 and d_N^2 suggest that in general d_N^r/c_N may be a simple rational function of N, with a denominator made up of factors like $N+2, N+3, \ldots, 4N-2, 4N-6, \ldots$. We find that this is true for sufficiently large N, and have empirically obtained d_N^r for $r = 1, \ldots, 12$.

Explicitly, for $r = 3$ and 4:

$$d_N^3 = -2Nc_N - 2c_N + 2c_{N+1} \tag{2.27}$$

$$d_N^4 = 3Nc_N + 2c_{N-1} - 5c_N + 5c_{N+1} - 2c_{N+2}. \tag{2.28}$$

More generally, it is convenient to define a double set of integers

$$c_{k,N} = \binom{N}{k} c_N = \frac{N!\,c_N}{k!(N-k)!} \tag{2.29}$$

(thus $c_{0,N} = c_N$ is the Nth Catalan number). Then

$$\begin{aligned}
&d_N^3 = -2c_{1,N} + e_N^3 \qquad d_N^4 = 3c_{1,N} + e_N^4 \qquad d_N^5 = -38c_{1,N-1} + 4c_{2,N} + e_N^5 \\
&d_N^6 = 13c_{1,N} - 4c_{2,N} + e_N^6 \qquad d_N^7 = -90c_{1,N-1} + 15c_{2,N} - 8c_{3,N} + e_N^7 \\
&d_N^8 = -745c_{1,N-2} - 5c_{2,N} + 4c_{3,N} + e_N^8 \\
&d_N^9 = 1947c_{1,N-1} + 51c_{2,N-1} - 24c_{3,N} + 16c_{4,N} + e_N^9 \\
&d_N^{10} = -121\,497c_{1,N-3} + 3566c_{2,N-2} - 19c_{3,N} + e_N^{10} \\
&d_N^{11} = 11\,065\,702c_{1,N-5} - 14\,441c_{2,N-2} + 6c_{3,N} + 40c_{4,N} - 32c_{5,N} + e_N^{11} \\
&d_N^{12} = -111\,135\,855c_{1,N-6} + 251\,284c_{2,N-3} - 588\tfrac{2}{5}c_{3,N} + 81\tfrac{3}{5}c_{4,N} - 16c_{5,N} + e_N^{12}
\end{aligned} \tag{2.30a}$$

where

$$e_N^r = \sum_{j=1}^{2r-4} \beta_j^r c_{N-r+2+j} \tag{2.30b}$$

and the coefficients β_j^r, for $r \geq 3$, are given in table 1.

Because of (2.26), there are various linear relations between $c_{0,N-1}, \ldots, c_{k,N-1}, c_{0,N}, \ldots, c_{k,N}$, for $k \geq 1$. This means that there are infinitely many ways of writing (2.30). As far as possible we have chosen forms that involve only

Table 1. The coefficients β_j^r in the extrapolation formula (2.30).

	r									
j	3	4	5	6	7	8	9	10	11	12
1	−2	2	2	6	28	168	1188	9 438	81 796	758 472
2	2	−5	−6	14	40	172	940	6 022	43 224	337 852
3		5	−20	58	76	230	916	4 358	22 626	116 670
4		−2	−2	−73	218	−23	1144	4 612	5 5531 158	−55 475 799
5			12	24	−360	1185	1609	−55 027	−1 358 232	14 008 377
6			−2	−6	174	−1619	5684	20 838	−317 870	3 608 148
7				0	−128	889	−7985	23 543	−52 903	900 703
8				0	82	−242	4320	−35 091	56 168	360 976
9					−15	−15	−1334	23 956	−177 599	405 215
10					0	18	−104	−8 722	124 506	−793 642
11						−1	243	2 820	−56 441	675 899
12						0	−46	−1 204	18 304	−329 150
13							−2	367	−5702	115 282
14							0	−44	1636	−25 924
15								2	−250	954
16								0	8	1036
17									−2	−125
18									0	−12
19										2
20										0

integer coefficients. The d'_N are all integers even though $\frac{1}{5}$ fractions occur in d^{12}_N: this is because $c_{3,N}+c_{4,N}$ is always divisible by five.

The formulae (2.30) are found to be true for all available values of N, provided N is sufficiently large that they involve the Catalan numbers c_m only for $m \geqslant 0$ (with $c_0 = 1$). For reasons that we do not understand, they are also true if they involve c_{-1}, provided we adopt the artificial convention

$$c_{-1} = 1. \tag{2.31}$$

Thus (2.30) is then valid for

$$N \geqslant r-4. \tag{2.32}$$

(We should like to have a corresponding convention for c_{-2}: this could certainly be done by adding terms to e^r_N proportional to $(4N'-2)c_{N'-1}-(N'+1)c_{N'}$, with $N' = N-r+5$. From (2.26) this vanishes for $N' \geqslant 1$, so only modifies e^r_N when terms $c_{-2}, c_{-3}, \ldots$, occur. However, we have found no obvious way of doing this.)

Using the extrapolation formulae (2.30) in (2.20), from the $N = 20$ results we can calculate $a_{N,m}$ for $N = 30, \ldots, 41$ and $m \leqslant 41$. From (2.19) we then have the series for the directed percolation probability $P(q)$ to order 41. The coefficients a_{NN} are given in table 2.

2.4. A transformation of the series

In the two-dimensional models that have been exactly solved, there is a natural parametrisation of the variables in terms of elliptic funcions (Baxter *et al* 1975, Baxter

Table 2. Coefficients a_{NN} and λ_N in $P(q)$ and $P(x)$, where $q = x-x^3-x^4+x^5$.

N	a_{NN}	λ_N	N	a_{NN}	λ_N
0	1	1	21	−16 782 444	7 186
1	0	0	22	−44 470 757	9 558
2	−1	−1	23	−118 090 648	21 800
3	−2	−2	24	−314 580 062	16 576
4	−4	−2	25	−839 379 548	61 234
5	−8	0	26	−2 245 969 278	−7 978
6	−17	2	27	−6 017 177 104	226 136
7	−38	4	28	−16 161 597 987	−446 034
8	−88	3	29	−43 448 897 414	1 118 180
9	−210	0	30	−117 083 094 891	−3 180 033
10	−511	−1	31	−315 709 399 172	6 428 640
11	−1264	6	32	−853 195 535 637	−24 607 343
12	−3165	28	33	−2 306 601 710 190	56 904 504
13	−8006	72	34	−6 249 350 665 825	−176 451 720
14	−20 426	139	35	−16 933 569 745 596	379 824 634
15	−52 472	242	36	−45 982 825 444 918	−1 133 204 742
16	−135 682	407	37	−124 847 185 166 968	3 043 377 452
17	−352 562	722	38	−339 715 065 397 631	−9 446 665 098
18	−920 924	1215	39	−923 984 791 735 474	25 288 847 986
19	−2 414 272	2348	40	−2 518 902 151 116 767	−68 663 095 038
20	−6 356 565	3753	41	−6 861 776 192 406 434	179 579 621 084

1982). The regularities in the series analysis of $P(q)$ suggested that this might have similar properties, so we tried expanding P and q as functions of a third variable x:

$$q=x\prod_{n=1}^{\infty}(1-x^n)^{q_n} \qquad P=P(q)=\prod_{n=1}^{\infty}(1-x^n)^{p_n}. \tag{2.33}$$

We looked for solutions in which the exponents p_n, q_n were all small integers with some periodicity property.

We failed to find such a solution, but we did note that if

$$q=x-x^3-x^4-x^5 \tag{2.34}$$

then the series for P as a function of x had coefficients four orders of magnitude smaller than the original series. We also give these reduced coefficients λ_N where

$$P=\sum_{N=0}^{\infty}\lambda_N x^N \tag{2.35}$$

in table 2: they at least provide a more compact way of presenting the information contained in the a_{NN}.

3. Analysis of series

Order parameter series are usually well suited to analysis by Dlog Padé approximants. Accordingly, we show in table 3 the standard Dlog Padé approximants, where the few defective approximants are marked with an asterisk. It can be seen that there is a slow but steady downward drift in the estimates of both q_c and the exponent β, until at about $N=14$ most entries are quite stable, suggesting a value of $q_c=0.355\,299$ and exponent $\beta=0.2764$. For both quantities the consistency of the latter estimates suggests error bars of 1 or 2 in the last quoted figure.

The rational number $\frac{199}{720}=0.276\,388\,888\ldots$ is adequately close to the exponent estimate β, and is the fraction that follows from scaling laws and the earlier estimates (Essam *et al* 1986) of the exponents γ, $\nu_\perp$ and $\nu_\parallel$. Furthermore, the critical percolation probability $p_c=1-q_c=0.644\,701$ is in precise agreement with that found by Essam *et al* (1986, 1988). However, if we accept this value of p_c and analyse the series using first-order inhomogeneous differential approximants, we obtain the results shown in

Table 3. Dlog Padé approximants to the percolation probability series for directed bond percolation on the square lattice. The entries give q_c (left) and β (right) estimates.

N	$[N-1/N]$		$[N/N]$		$[N+1/N]$	
9	0.355 317	0.276 8	0.355 317	0.276 8	0.355 308	0.276 5
10	0.355 306	0.276 6	0.355 310*	0.276 7	0.355 302	0.276 51
11	0.355 300	0.276 45	0.355 303	0.276 5	0.355 3023	0.276 65
12	0.355 3016	0.276 49	0.355 3011	0.276 48	0.355 2997	0.276 44
13	0.355 3028*	0.276 52	0.355 3004	0.276 46	0.355 3000	0.276 45
14	0.355 2995	0.276 43	0.355 2972	0.276 34	0.355 2995	0.276 43
15	0.355 2991	0.276 42	0.355 2995	0.276 43	0.355 2995	0.276 43
16	0.355 2995	0.276 43	0.355 2995	0.276 43	0.355 2995	0.276 43
17	0.355 2995	0.276 43	0.355 2995	0.276 43	0.355 2997*	0.276 44
18	0.355 2994	0.276 43				

table 4. These give rise to the estimate $\beta = 0.2770 \pm 0.0003$, where the confidence limits quoted are three times the apparent scatter in the estimates of β. This estimate nevertheless just excludes that given by the Padé approximants (which are just homogeneous first-order differential approximants) and is closer to the much simpler fraction $\frac{5}{18} = 0.2777\ldots$, which has the additional appeal that it mirrors the denominator conjectured for γ, that is, $\gamma = \frac{41}{18}$. Together, these then give rise to the estimate $\alpha = -\frac{5}{6}$ and $\delta = \frac{46}{5}$.

We are thus in the situation that two different methods of series analysis gives rise to two slightly different exponent estimates. Considering the two methods, it is clear that the simpler Padé method must be preferred. The basic rule of series analysis is that the method used should represent the underlying functional form as closely as possible. Inhomogeneous differential approximants are appropriate for functions which behave like

$$f(x) \sim A(x) + B(x)(1 - x/x_c)^{\beta} \tag{3.1}$$

Table 4. First-order inhomogeneous differential approximants biased at $q_c = 0.355\,299$ giving estimates of β. Entries to the right of the table use most series coefficients. Entries are values of $[L/N-1;\ N]$, $[L/N;\ N]$ and $[L/N+1;\ N]$, reading from top to bottom.

	N										
L	9	10	11	12	13	14	15	16	17	18	19
1	0.2690	0.2747	0.2737	0.2743	0.2754	0.2757	0.2756	0.2765	0.2765	0.2764	0.2764
	0.2716	0.2734	0.2724	0.2757	0.2755	0.2757	0.2752	0.2767	0.2763	0.2764	0.2764
	0.2752	0.2737	0.2745	0.2754	0.2757	0.2756	0.2766	0.2766	0.2764	0.2765	0.2764
2	0.2796	0.2671	0.2765	0.2768	0.2777	0.2759	0.2773	0.2780	0.2774	0.2764	0.2764
	0.2948	−1.2162	0.2794	0.2763	0.2762	0.2812	0.2772	0.2777	0.2770	0.2761	0.2770
	0.2624	0.2766	0.2768	0.2779	0.2759	0.2773	0.2738	0.2774	0.2766	0.2761	0.2771
3	0.2819	0.2724	0.2791	0.2782	0.2784	0.2772	0.2773	0.2774	0.2771	0.2759	0.2771
	0.2789	0.2784	0.2790	0.2781	0.2765	0.2773	0.2773	0.2774	0.2771	0.2769	0.2770
	0.2698	0.2791	0.2781	0.2782	0.2772	0.2773	0.2773	0.2774	0.2747	0.2771	
4	0.2789	0.3058	0.2776	0.2781	0.2773	0.2768	0.2776	0.2772	0.2647	0.2647	0.2770
	0.2787	0.2504	0.2751	0.2764	0.2768	0.2773	0.2773	0.2784	0.2767	0.2772	
	0.2903	0.2777	0.2781	0.2773	0.2773	0.2779	0.2776	0.2772	0.2787	0.2770	
5	0.2749	0.2776	0.2770	0.2769	0.2768	0.2770	0.2776	0.2770	0.2769	0.2769	
	0.2744	0.2767	0.2775	0.2769	0.2768	0.2779	0.2765	0.2769	0.2769	0.2770	
	0.2766	0.2770	0.2769	0.2768	0.2769	0.2779	0.2771	0.2769	0.2769		
6	0.2743	0.2765	0.2787	0.2768	0.2769	0.2774	0.2766	0.2769	0.2769	0.2768	
	0.2744	0.2771	0.2773	0.2768	0.2768	0.2771	0.2760	0.2769	0.2769		
	0.2767	0.2339	0.2768	0.2769	0.2773	0.2766	0.2770	0.2769	0.2767		
7	0.3001	0.2774	0.2769	0.2769	0.2778	0.2748	0.2769	0.2769	0.2769		
	0.2771	0.2776	0.2777	0.2776	0.2771	0.2769	0.2769	0.2769	0.2769		
	0.2276	0.2770	0.2767	0.2777	0.2775	0.2769	0.2769	0.2769			
8	0.2770	0.2776	0.2754	0.2745	0.2769	0.2769	0.2769	0.2769	0.2769		
	0.2773	0.2773	0.2773	0.2843	0.2769	0.2768	0.2769	0.2769			
	0.2776	0.2766	0.2595	0.2769	0.2769	0.2769	0.2769	0.2769			
9	0.2777	0.2773	0.2769	0.2720	0.2769	0.2769	0.2769	0.2769			
	0.2776	0.2774	0.2777	0.2768	0.2769	0.2771	0.2769	0.2769			
	0.2773	0.2770	0.4090	0.2769	0.2769	0.2769	0.2769				
10	0.2776	0.2756	0.2782	0.2770	0.2769	0.2768	0.2769	0.2766			
	0.2800	0.2770	0.2774	0.2769	0.2768	0.2768	0.2769				
	0.2765	0.2775	0.2770	0.2769	0.2768	0.2769	0.2776				

where A and B are regular in the neighbourhood of x_c. For the order parameter, the percolation probability, $A(x_c) = 0$. Such special cases of (3.1) are precisely the form assumed by the Dlog Padé method. The only complicating feature which may be present is that of confluent non-analytic terms—corresponding to B being non-regular in the vicinity of x_c. Now since $A(x_c) = 0$, we can test this possibility by transforming the series using the transformation devised by Baker and Hunter (1973). Assuming x_c is known, or, as in this case, accurately estimated, we make the substitution

$$x = x_c[1 - \exp(-\xi)]. \tag{3.2}$$

This produces a power series in ξ. Multiplying the coefficient of ξ^k by $k!$ gives an auxiliary function $F(\xi)$ with the poles located at points equal to the reciprocal of critical exponents of f.

We show estimates of the two largest exponents for a range of diagonal and near-diagonal approximants in table 5. There it can be seen that the two entries are converging to 3.615 and 0.785 or thereabouts. The reciprocal of these values gives the dominant and subdominant exponents, which are therefore 0.2766 and 1.274, respectively; that is, a correction to scaling exponent of 1.003 (neglecting any uncertainties in the exponent values). This strongly suggests that the leading correction term is analytic. Thus the order parameter appears to be of precisely the functional form (3.1) with $A(x_c) = 0$ that underlies the Dlog Padé method, and so we must favour the results from this method.

Table 5. Padé approximants to transformed order parameter series assuming $q_c = 0.355\,299$. The entries give the reciprocal of the dominant (left) and subdominant (right) exponents.

N	$[N-1/N]$		$[N/N]$		$[N+1/N]$	
15	3.6199	0.0843	3.6097	0.7920	3.4907	0.8204
16	3.6179	0.7808	3.6142	0.7866	3.5610	0.8020
17	3.6180	0.7793	3.6090	0.7919	3.5608	0.8020
18	3.6180	0.7794	3.6153	0.7850	3.6207	0.7831
19	3.6171	0.7837	3.6153	0.7851	3.6151	0.7851
20	3.6151	0.7851	3.6153	0.7851	3.6147	0.7854
21	3.6141	0.7854				

Recourse to other methods of series analysis, such as the wide range of ratio type methods that exist, is found to be unsatisfactory. Because q_c is not the closest singularity to the origin, these methods cannot be directly used. A range of transformations have also been tried, but these do not produce exponent estimates of sufficient accuracy. Indeed, the estimates of β obtained by these methods is typically 0.275 ± 0.001.

Accordingly, we conclude that $\beta = 0.2764 \pm 0.0001$, and that the fraction $\frac{199}{720}$ is the most likely exact rational. That being said, the appeal of the slightly larger fraction $\frac{5}{18} = 0.2777\ldots$ is undeniably far greater aesthetically. Regrettably, we are unable to find any convincing numerical evidence to favour this value.

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